Journal of Computational Physics **163,** 262–267 (2000)
doi:10.1006/jcph.2000.6561, available online at http://www.idealibrary.com on IDEAL®

NOTE

On Newton–Krylov Multigrid Methods for the Incompressible Navier–Stokes Equations

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Received October 11, 1999; revised June 12, 2000

In this note we continue our investigation [1] of multigrid methods as preconditioners to a Jacobian-free Newton–Krylov method [2, 3]. We consider two different options for the formation of the coarse grid operators required in the multigrid preconditioner. The first option (Method 1) involves restricting the dependent variables down through a series of grids, rediscritizing, and forming the coarse grid matrices. In the second option, considered (Method 2), the coarse grid matrices are formed from the fine grid matrix using a Galerkin [4] procedure. Both methods use low-complexity piecewise constant restriction and prolongation. Additionally, we consider the option of using either a coupled or a distributed (segregated) [4, 5] approach in our preconditioner. In the more standard coupled approach, the multigrid smoother works directly on the coupled system of equations. In the distributed approach, each equation in the system is treated individually in the preconditioner and approximately inverted using a scalar multigrid approach. We use the standard driven cavity problem [6] and the natural convection problem [7] as our test problems. Other research in this area includes the work of Pernice [8], who is studying combinations of SIMPLE [9], nonlinear multigrid, and Newton–Krylov methods.

Newton's method requires the solution of the linear system

$$\mathbf{J}^n \delta\mathbf{u}^{n+1} = -\mathbf{F}(\mathbf{u}^n), \qquad \mathbf{u}^{n+1} = \mathbf{u}^n + d\delta\mathbf{u}^{n+1}, \tag{1}$$

where $\mathbf{J}$ is the Jacobian matrix, $\mathbf{F}(\mathbf{u})$ is the nonlinear system of equations (the discretized partial differential equations), $\mathbf{u}$ is the state vector, $\delta\mathbf{u}$ is the Newton update vector, d is an adaptively chosen damping scalar, and n is the Newton iteration level. The Generalized Minimal RESidual (GMRES) algorithm [10] is used to solve Eq. (1). The GMRES algorithm requires the action of the Jacobian ($\mathbf{J}$) only in the form of Jacobian-vector products, which may be approximated by a first-order Taylor series expansion [2, 3],

$$\mathbf{J}\mathbf{v} \approx [\mathbf{F}(\mathbf{u} + \epsilon\mathbf{v}) - \mathbf{F}(\mathbf{u})]/\epsilon, \tag{2}$$

0021-9991/00 $35.00

where $\mathbf{v}$ is a Krylov vector (part of the GMRES iteration), and ϵ is a small perturbation. The use of an iterative technique to solve Eq. (1) does not require the exact solution of the linear system as discussed in [1]. The right preconditioned version of Eq. (2) is

$$\mathbf{J}\mathbf{P}^{-1}\mathbf{v} \approx [\mathbf{F}(\mathbf{u} + \epsilon\mathbf{P}^{-1}\mathbf{v}) - \mathbf{F}(\mathbf{u})]/\epsilon. \tag{3}$$

Only the matrix (or matrices) which is required for $\mathbf{P}^{-1}$, the preconditioning process, is formed. There are two choices to be made: the linearization to be used to form the matrices required in $\mathbf{P}^{-1}$ and the linear iterative method to be used for the action of $\mathbf{P}^{-1}$. For the second choice we will use multigrid as in [1, 5].

An option available to the Jacobian-free method is the use of distributed relaxation (a segregated solver) as a preconditioner,

$$\mathbf{J}\mathbf{P}^{-1}\mathbf{v} \approx \frac{\mathbf{F}_{\text{coupled}}\left(\mathbf{u} + \epsilon\mathbf{P}_{\text{dist}}^{-1}\mathbf{v}\right) - \mathbf{F}_{\text{coupled}}(\mathbf{u})}{\epsilon}. \tag{4}$$

Here, $\mathbf{F}_{\text{coupled}}(\mathbf{u})$ denotes the nonlinear function evaluated fully coupled (as is always the case), and $\mathbf{P}_{\text{dist}}^{-1}$ denotes the preconditioning process handled in a distributed manner. We have used a distributed solution approach as a preconditioner for the Jacobian-free method on a system of time-dependent reaction diffusion equations [5]. In this paper we will demonstrate the capability on a steady-state convection diffusion system.

The two-dimensional incompressible Navier–Stokes equations, in stream function–vorticity (ψ, ω) formulation are

$$\nabla^2\psi = \omega, \tag{5}$$

$$\nabla \cdot (\vec{V}\omega) - \frac{1}{\text{Re}}\nabla^2\omega = 0, \tag{6}$$

with $\vec{V} = V_1\hat{x} + V_2\hat{y}$, and $V_1 = \frac{\partial\psi}{\partial y}$, $V_2 = -\frac{\partial\psi}{\partial x}$. The standard driven cavity [6] in the unit square is used as the first model problem. A finite volume discretization is used with second-order [11] upwind differencing for the convective operator and second-order central differencing for the diffusion operators. First-order upwinding is used in the preconditioner [12], and the advecting velocity is treated as known in the preconditioner. All simulations are started on a 10×10 grid and a mesh sequencing algorithm [1] is used to move up to a 320×320 grid. The performance is compared using the two different coarse grid operators, Method 1 and Method 2, both with the coupled preconditioner. The coupled multigrid preconditioner uses a block version of symmetric Gauss–Seidel (SGS) as a smoother. The block is a 2×2, coupling ψ and ω in each finite volume. The preconditioner consists of one V-cycle with an equal, and fixed, number of pre- and post-smoothing sweeps. The nonlinear convergence tolerance is $[\sum_{i=1}^{N_{dim}}(F_{\psi,i})^2 + (F_{\omega,i})^2]^{\frac{1}{2}} < 10^{-4}$, where $F_{\psi,i}$ and $F_{\omega,i}$ are the nonlinear residuals at grid cell i, and there are $N\,dim$ grid cells. The linear convergence tolerance (at each Newton iteration) is $\|\mathbf{J}^n\delta\mathbf{u}^{n+1} + \mathbf{F}(\mathbf{u}^n)\|_2 < \gamma\|\mathbf{F}(\mathbf{u}^n)\|_2$, with $\gamma = 5 \times 10^{-2}$.

Table I presents results for for $\text{Re} = 1000$. In the second column the numbers 2–1 mean 2 smoothing passes (pre and post) and 1 V-cycle. All CPU times are on an SGI Octane. There is a clear performance advantage for Method 1, although Method 2 is always competitive. This is a correction to earlier results presented in [13], which were in error by giving the advantage to Method 2. We see that for this problem, additional smoothing (4–1) improves the performance of Method 2.

TABLE I
Performance of Method 1 and Method 2 (Coupled Smoother) for Re = 1000

Method	Multigrid option smooth-V	Grid 4 80 × 80		Grid 5 160 × 160		Grid 6 320 × 320		CPU time (s)
		N	K/N	N	K/N	N	K/N	
Method 1	2-1	4	5.00	4	3.75	4	2.5	91
Method 1	4-1	4	5.25	4	3.75	3	2.66	123
Method 2	2-1	5	7.80	4	6.50	4	12.00	264
Method 2	4-1	4	5.50	4	5.00	4	3.00	162

Note. N = Newton iterations; K/N = Krylov iterations per Newton iteration.

To further study the performance of our proposed methods we will solve the two-dimensional incompressible Navier–Stokes plus energy equations, in stream function–vorticity–temperature (ψ, ω, T) formulation, with the Boussinesq approximation,

$$\nabla^2 \psi = \omega, \tag{7}$$

$$\nabla \cdot (\vec{V}\omega) - \nabla^2 \omega = \mathrm{Gr} \nabla_x T, \tag{8}$$

$$\nabla \cdot (\vec{V} T) - \frac{1}{\mathrm{Pr}} \nabla^2 T = 0. \tag{9}$$

Gr is the Grashof number, Pr is the Prandtl number (0.7 is used here), and the gravity vector is in the negative y direction. The standard natural convection problem [7] in the unit square is used as the second model problem. All simulations are started on a 10×10 grid and a mesh sequencing algorithm is used to move up to a 320×320 grid. Damping is applied to limit the maximum relative change in the temperature, in any given finite volume, to no more than 50% per Newton iteration. The coupled multigrid preconditioner is analogous to that of the previous problem and uses a block version of symmetric Gauss–Seidel. For the natural convection problem the block is a 3×3, coupling ψ, ω, and T in each finite volume. The preconditioner consists of one V-cycle with equal, and fixed, number of pre- and post-smoothing sweeps. The nonlinear convergence tolerance is a normalized measure $\|\mathbf{F}(\mathbf{u}^n)\|_2 / \|\mathbf{F}(\mathbf{u}^0)\|_2 < 10^{-4}$, and the linear convergence tolerance is $\|\mathbf{J}^n \delta \mathbf{u}^{n+1} + \mathbf{F}(\mathbf{u}^n)\|_2 < 5 \times 10^{-2} \|\mathbf{F}(\mathbf{u}^n)\|_2$.

On the driven cavity problem, we have used a simple, low-complexity coupled preconditioner. Instead, in the preconditioner, one could approximate the inverse of each equation independently. This would be a "distributed relaxation" preconditioner, and would require less memory and possibly less CPU time per iteration. In each of the three separate elliptic solves we will use a scalar version of the proposed multigrid method with one V-cycle and an SGS smoother. The only complication to this approach is that the boundary conditions on Eq. (8) are a strong function of ψ^{n+1}, and the source term in Eq. (7) is ω^{n+1}. This complication is addressed by using a 2×2 linear solve, to couple ψ and ω in the preconditioner, only on the fine grid. In the distributed preconditioner the approximate inversion of Eq. (9) is done first (one multigrid V-cycle) and then the 2×2 linear solve, and then Eqs. (7) and (8) are approximately inverted (one multigrid V-cycle each), followed by one more pass of the 2×2 linear solve.

Table II presents results for Method 1 and Method 2, for both the coupled and the distributed preconditioner, at $\mathrm{Gr} = 1 \times 10^5$. Both Method 1 and Method 2 produce a small

TABLE II
Comparison of Method 1 and Method 2 for for Gr = 1×10^5, Coupled and Distributed Smoother

Coarse grid method	Smoothing option	Grid 4 80×80		Grid 5 160×160		Grid 6 320×320		CPU time (s)
		N	K/N	N	K/N	N	K/N	
Method 1	Coup.	4	2.0	3	1.67	3	1.67	82
Method 2	Coup.	4	5.0	3	5.67	2	1.0	77
Method 1	Dist.	3	5.67	3	6.33	3	9.0	137
Method 2	Dist.	4	6.5	3	6.33	3	5.3	106

Note. N = Newton iterations; K/N = Krylov iterations per Newton iteration.

number of GMRES iteration per Newton iteration. It can be seen that the distributed preconditioner may require more GMRES iterations per Newton iterations; however, due to its low computational complexity (scalar elliptic problems) it is always competitive in terms of CPU time. It appears that Method 2 is to be preferred for this problem.

Figure 1 is a plot of CPU time scaling, as a function grid dimension for Method 2, using both the distributed and the coupled approaches, $\mathrm{Gr} = 1 \times 10^5$. The data are for 80×80, 160×160, and 320×320 problems. We include a reference line for linear scaling, and we see that both approaches scale better than linear.

Table III compares the performances of Methods 1 and 2, coupled and distributed, for $\mathrm{Gr} = 1.0 \times 10^7$. This is a challenging boundary value problem and thus we use a continuation process in which $\mathrm{Gr} = 5.0 \times 10^6$ on the 80×80 grid, $\mathrm{Gr} = 7.5 \times 10^6$ on the 160×160 grid, and $\mathrm{Gr} = 1.0 \times 10^7$ on the 320×320 grid. With this continuation process all four approaches could obtain the solution. We see that Method 2 is preferred here. The coupled approach results in the minimum GMRES iterations while the distributed approach produces the minimum CPU time.

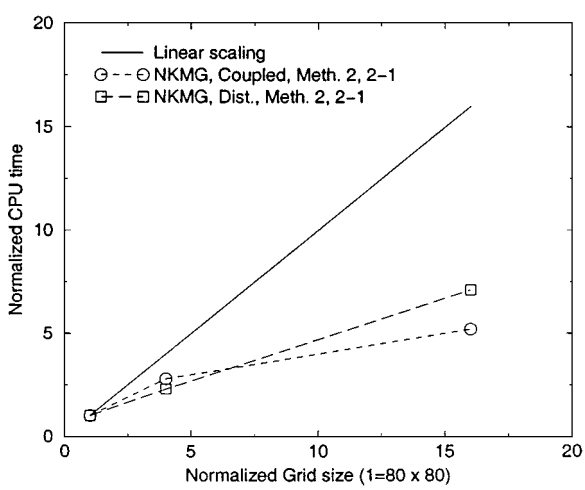

FIG. 1. Scaling (CPU time vs problem size) of Newton–Krylov multigrid (NKMG) for coupled and distributed preconditioners, Method 2, for $\mathrm{Gr} = 1.0 \times 10^5$.

TABLE III
Comparison of Method 1 and Method 2 for for Gr = 1×10^7, Coupled and Distributed Smoother

Coarse grid method	Smoothing option	Grid 4 80×80		Grid 5 160×160		Grid 6 320×320		CPU time (s)
		N	K/N	N	K/N	N	K/N	
Method 1	Coup.	5	12.80	6	14.50	7	18.71	847
Method 2	Coup.	4	8.75	4	10.50	4	15.75	464
Method 1	Dist.	6	18.83	5	21.80	5	26.40	580
Method 2	Dist.	4	9.00	4	14.25	4	24.75	446

Note. $N =$ Newton iterations; $K/N =$ Krylov iterations per Newton iteration.

We have presented a new nonlinear multilevel iterative method based on multigrid preconditioned, Jacobian-free Newton–Krylov methods. Performance aspects of this new algorithm have been demonstrated on two 2-D incompressible Navier–Stokes problems. Specifically we have considered different options for forming the coarse grid matrices required in the multigrid preconditioner and both coupled and distributed smoothers. It has been demonstrated that simple linearizations and low-complexity multigrid methods can make effective preconditioners for the Jacobian-free Newton–Krylov method. Method 1 and Method 2 are both viable options as are the coupled and distributed approaches.

ACKNOWLEDGMENT

This work was performed under the auspices of the U.S. Department of Energy under DOE Contract W-7405-ENG-36 at Los Alamos National Laboratory.

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